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Term structure models

(work in progress)

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1 Definitions

Let $P(t,u)$ be the price of a zero coupon bond at time t maturing at time u. Define the *spot rate* (yield to maturity) such that

$$\begin{aligned} P(t,u) &= e^{-r(t,u)\cdot(u,t)} \\ r(t,u) &= -\frac{\log P(t,u)}{u-t} \end{aligned}$$

Define the *forward rate* such that

$$\begin{aligned} P(t,u) &= e^{-\int_t^u f(t,s)ds} \\ f(t,u) &= \frac{\partial(-\log P(t,u))}{\partial u} \end{aligned}$$

Define the *short rate* as the "rate for overnight borrowing"

$$\begin{aligned} r(t) &= \lim_{u\to t} r(t,u) \\ &= f(t,t) \\ P(t,u) &= \mathbb{E}^Q\left[e^{-\int_t^u r(s)ds}\right] \end{aligned}$$

where Q is the risk neutral measure.

The absence of arbitrage opportunities implies the existence of a stochastic discount factor (*state price density* or *pricing kernel* ρ) so that the price of any contingent claim X at time 0 is

$$\mathbb{E}[\rho_T \cdot X] = \int_{\text{states of the world}} X(\omega)\cdot\rho_T(\omega)\,dP(\omega)$$

where P is the P-probability.

Now define the Q-probability $dQ(\omega)$ of a state as

$$dQ(\omega) = \underbrace{e^{-\int_0^T r(s)ds}}_{>0}\cdot\underbrace{\rho_T(\omega)}_{>0}\cdot\underbrace{dP(\omega)}_{>0} \tag{1.1}$$

The Q-probability has the following properties:

- $dQ(\omega) > 0 \qquad \forall\omega$

- $Q\,(\text{sure event}) = 1$

Example:

Consider the following strategy: put one unit of currency in the bank at time 0. At time T you will have $X = e^{\int_0^T r(s)ds}$. If the price of X at time 0 is 1 it follows

$$\begin{aligned} 1 &= \mathbb{E}\left[\rho_T \cdot X\right] \\ &= \int_{\text{states}} X(\omega) \cdot \rho(\omega)\, dP(\omega) \\ &= \int_{\text{states}} e^{\int_0^T r(s)ds} \cdot \rho_T(\omega)\, dP(\omega) \\ &= \int_{\text{states}} dQ(\omega) \end{aligned}$$

The term $\frac{dQ}{dP}$ is called the *Radon-Nikodyn derivative*.

Definition: The price of any contingent claim X at time 0 is

$$\mathbb{E}^Q\left[e^{-\int_0^T r(s)ds} \cdot X\right]$$

We discount with the short rate.

Proof: The price of X is

$$\int_{\text{states}} \rho_T(\omega) \cdot X(\omega)\, dP(\omega)$$

From formula 1.1 we know that

$$\rho_T(\omega) = \frac{dQ}{dP} \cdot e^{-\int_0^T r(s)ds}$$

So we can calculate

$$\begin{aligned} \int_{\text{states}} \rho_T(\omega) \cdot X(\omega)\, dP(\omega) &= \int_{\text{states}} \frac{dQ}{dP} \cdot e^{-\int_0^T r(s)ds} \cdot X(\omega)\, dP(\omega) \\ &= \int_{\text{states}} e^{-\int_0^T r(s)ds} \cdot X(\omega)\, dQ \end{aligned}$$

So we need a model of $r(t)$ under Q; this is called a *term structure model*.

2 Term structure models

2.1 Vasicek

Under Q the model suggest

$$dr(t) = \kappa \cdot (\theta - r(t))\,dt + \sigma dW(t)$$

where κ, θ, and σ are constants and W is a brownian motion under Q ($W(t) - W(s) \sim N[0, t-s] \quad \forall t > s$). The constant θ denotes the long run mean of the short rate and $\kappa > 0$ the rate of mean reversion.

To find $P(t,u)$ we need to know the distribution of $\int_t^u r(s)\,ds$ given $r(t)$. So the bond price is a function of the short rate, but ..... distribution of the short rate at two different time points and intervals are the same for the same starting point, i.e. if $r(t) = r(t')$ then the distribution of $\int_t^u r(s)\,ds$ is the same as the distribution of $\int_{t'}^{u'} r(s)\,ds$. So $P(t,u)$ is a functin of $r(t)$ and $u - t$, but which function?

2.2 Cox-Ingersoll-Ross

This model was introduced 1985 and is also called the *Square-Root model*. The short rate is defined by

$$dr(t) = \kappa \cdot (\theta - r(t))\,dt + \sigma \cdot \sqrt{r(t)}dW(t) \tag{2.1}$$

where κ, θ, σ and W have the same meaning as in the Vasicek model. This process is like in the Vasicek model a Markov process.

2.3 Comparison of Vasicek and CIR

The two models of Vasicak and CIR differ in the following way:

In the Vasicek model

- Given $r(t)$ and $s > t$, $r(s)$ is normally distributed.
- Given $r(t)$ the term $\int_t^u r(s)\,ds$ is normally distributed.

- Negative values of the short rate are possible.
- The short rate has constant volatility and normally distributed increments

$$r(s) = e^{-\kappa\cdot(\theta-r)}\cdot(\theta - r(t)) + r(t) + \underbrace{\int \ldots dW}_{\text{normally distributed}}$$

In the CIR model

- The short rate is always non-negative.
- The volatility vanishes while going to zero; only the drift is left.

Both models are affine one-factor models. “One factor” means that bond prices and in particular $\int_t^u r(s)\,ds$ at time t depends only on a single variable, namely $r(t)$. “Affine” means that the drift coefficient and the variance (square of the dW coefficient) are affine, i.e. linear and constant, functions of the state variable.

The general affine one-factor model is

$$dr = \kappa\cdot(\theta - r)\,dt + \sqrt{a + b\cdot r}dW$$

Vasicek is one special case with $b = 0$ and CIR is another special case with $a = 0$.

In affine models yields $r(t,u)$ are affine functions of the state variables. In an affine one-factor model we have

$$\begin{aligned} P(t,u) &= e^{-r(t,u)\cdot(u-t)} \\ &= e^{-\tau\cdot a(\tau)-\tau\cdot b(\tau)\cdot r(t)} \end{aligned}$$

where

$$\begin{aligned} r(t,u) &= a(\tau) + b(\tau)\cdot r(t) \\ \tau &= u - t \end{aligned}$$

We call τ the remaining time or the time left to maturity.

2.4 Solution of Vasicek

The Vasicek solution for f is given by

$$\begin{aligned} P(t,u) &= \mathbb{E}^Q\left[e^{-\int_t^u r(s)ds}\middle| r(t)\right] \\ \int_t^u r(s)\,ds &\sim N\left[\text{function of } \tau \text{ and } r(t),\ \text{function of } \tau\right] \end{aligned}$$

so the mean is an affine function of $r(t)$. We can write

$$P(t,u) = e^{-\text{mean of } \int_t^u r(s)ds + \frac{1}{2}\cdot\text{variance of } \int_t^u r(s)ds}$$

2.5 Dai and Singleton: Specification analysis of affine term structure models

Generally the short rate of interest is defined by[1]

$$\begin{aligned} r(t) &= \delta_0 + \delta' \cdot Y(t) \\ dY(t) &= \kappa \cdot (\theta - Y(t))\, dt + \Sigma \cdot \sqrt{S(t)} dW(t) \end{aligned}$$

where Y is a N-vector of factors and W is a vector of independent standard Brownian motions under the risk neutral measure Q.

$$S = \text{diagonal} = \begin{pmatrix} \alpha_1 + \beta_1' \cdot Y(t) & & 0 \\ & \ddots & \\ 0 & & \alpha_N + \beta_N' \cdot Y(t) \end{pmatrix}$$

We can calculate the discount bond prices as

$$\begin{aligned} P(t,u) &= \mathbb{E}^Q \left[e^{-\int_t^u r(s) ds} \middle| Y(t) \right] \\ P(t,u) &= e^{-A(u-t) - B(u-t) \cdot Y(t)} \end{aligned}$$

where A and B are functions of the time to maturity. Further we have

$$\begin{aligned} dY_i(t) &= \sum_{l=1}^{N} \kappa_{il} \cdot (\theta_l - Y_l(t))\, dt + \sum_{l=1}^{N} \sigma_{il} \cdot \sqrt{\alpha_l + \beta_l' \cdot Y(t)} dW_l(t) \\ (dY_i(t))^2 &= \sum_{l=1}^{N} \sigma_{il}^2 \cdot \left(\alpha_l + \beta_l' \cdot Y(t) \right) dt \end{aligned}$$

If Y_j can be negative then $\beta_{lj} = 0$ for all l.

2.5.1 Example: Two factor model

Suppose a two-factor model

$$\begin{aligned} dY_1 = {} & \kappa_{11} \cdot (\theta_1 - Y_1)\, dt + \kappa_{12} \cdot (\theta_2 - Y_2)\, dt + \\ & + \sigma_{11} \cdot \sqrt{\alpha_1 + \beta_{11} \cdot Y_1 + \beta_{12} \cdot Y_2} dW_1 + \sigma_{12} \cdot \sqrt{\alpha_2 + \beta_{21} \cdot Y_1 + \beta_{22} \cdot Y_2} dW_2 \end{aligned}$$

Let $\alpha_1, \alpha_2 \geq 0$.

- If $\sigma_{11} \neq 0$ and $\alpha_1 > 0$, then Y_1 can be negative.

[1] The symbol Σ is a parameter and does *not* indicate a sum.

- If $\sigma_{12} \neq 0$ and $\alpha_2 > 0$, then Y_1 can be negative.

To ensure that Y_1 is always non-negative it must be

$$\begin{aligned} dY_1 &= \kappa_{11} \cdot (\theta_1 - Y_1)\, dt + \kappa_{12} \cdot (\theta_2 - Y_2)\, dt + \\ &+ \sigma_{11} \cdot \sqrt{\alpha_1 + \beta_{11} \cdot Y_1 + \beta_{12} \cdot Y_2} dW_1 + \sigma_{12} \cdot \sqrt{\alpha_2 + \beta_{21} \cdot Y_1 + \beta_{22} \cdot Y_2} dW_2 \end{aligned}$$

and more

$$\begin{aligned} \kappa_{12} &\leq 0 \\ \kappa_{11} \cdot \theta_1 &> 0 \end{aligned}$$

to get a positive drift and either

$$\sigma_{11} \cdot \beta_{11} \neq 0$$

or

$$\sigma_{12} \cdot \beta_{21} \neq 0$$

2.6 Balduzzi-Das-Foresi-Sundaram three factor model

This model is using three factors

$$\begin{aligned} (r, v, \theta) &= Y && (2.2) \\ dr &= n \cdot (\theta - r)\, dt + \sqrt{v} dW_1 && (2.3) \\ dv &= \gamma \cdot (\mu - v)\, dt + \sigma \cdot \sqrt{v} dW_2 && (2.4) \\ d\theta &= \lambda \cdot (\eta - \theta)\, dt + \phi dW_3 && (2.5) \end{aligned}$$

We see that r can become negative, v can not because it is as square root process and $d\theta$ is a gaussian process. In this model θ specifies a random long-run mean.

2.7 Lin-Chen three factor model

This model is also based on the formulas 2.3 and 2.4, but uses the following formula instead of 2.5

$$d\theta = \lambda \cdot (\eta - \theta)\, dt + \phi \cdot \sqrt{\theta} dW_S \qquad (2.6)$$

which is a square root process, i.e. now θ is also positive. If we now plug in formula 2.6 into formula 2.3 we get

$$\sqrt{a + b \cdot \theta + c \cdot v} \rightarrow \sqrt{v}$$

2.8 Covariance

$$\begin{aligned}
(dY_i)(dY_j) &= \sum_{l=1}^{N} \sigma_{il} \cdot \sigma_{jl} \cdot \left(\alpha_l + \beta_l' \cdot Y'(t)\right) dt \\
(dY) \cdot (dY)' &= \Sigma \cdot \sqrt{S} \cdot dW \cdot (dW)' \cdot \sqrt{S} \cdot \Sigma' \\
&= \Sigma \cdot S(t) \cdot \Sigma' \\
dY &= \ldots dt + \Sigma \cdot \sqrt{S(t)} dW \\
(dW) \cdot (dW)' &= I_{NxN} dt
\end{aligned}$$

Fundamental PDE

Now fix a maturity date u and write

$$\begin{aligned}
f(t, Y(t)) &= \underbrace{P(t,u)}_{\text{stochastic process}} \qquad f : \mathbb{R}^{N+1} \to \mathbb{R} \\
&= \text{price of discount bond maturing at date } u
\end{aligned}$$

Under Q we have

$$\frac{dP}{P} = r dt + \text{stochastic part}$$

To compute the drift use the fact that

$$f(t, Y) = e^{-A(u-t) - B(u-t)' \cdot Y}$$

Now set

$$Z = -A(u-t) - B(u-t)' \cdot Y$$

so the bond price is

$$P(t, u) = e^{Z}$$

Now apply Ito's Lemma

$$\begin{aligned}
\frac{dP}{P} &= dZ + \frac{1}{2} \cdot (dZ)^2 \\
dZ &= \dot{A}(u-t) dt + \nabla B' \cdot Y dt - B(u-t)' dY \\
\nabla B &= \begin{pmatrix} \dot{B}_1 \\ \vdots \\ \dot{B}_N \end{pmatrix} \\
(dZ)^2 &= B(u-t)' \cdot (dY) \cdot (dY)' \cdot B(u-t) \\
&= B(u-t)' \cdot \Sigma \cdot S(t) \cdot \Sigma \cdot B(u-t) \\
\frac{dP}{P} &= \dot{A} dt + \nabla B' \cdot Y dt - B' \left[\kappa \cdot (\theta - Y) dt + \Sigma \cdot \sqrt{S} dW\right] + \frac{1}{2} \cdot B' \cdot \Sigma \cdot S \cdot \Sigma' dt
\end{aligned}$$

As the drift of this process must be equal to the short rate we have

$$\dot{A} + \nabla B' \cdot Y - B' \cdot \kappa \cdot (\theta - Y) + \frac{1}{2} \cdot B' \cdot \Sigma \cdot S \cdot \Sigma' \cdot B \quad = \quad \underbrace{\delta_0 + \delta' \cdot Y}_{r} \tag{2.7}$$

Further

$$\begin{aligned} S &= \begin{pmatrix} \alpha_1 + \beta_1' \cdot Y & & 0 \\ & \ddots & \\ 0 & & \alpha_N + \beta_N' \cdot Y \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_N \end{pmatrix} + \begin{pmatrix} \beta_1' \cdot Y & & 0 \\ & \ddots & \\ 0 & & \beta_N' \cdot Y \end{pmatrix} \end{aligned}$$

Now we do

1. Match the coefficient of Y with δ
2. Match the constant on left with δ_0

We have $N+1$ ordinary differential equations. As boundary condition we have

$$\begin{aligned} P(u,u) &= e^{-A(0)-B(0)'\cdot Y} \\ &= 1 \qquad \forall Y \\ A(0) &= 0 \\ B(0) &= 0 \end{aligned}$$

First solve 1(so called *Riccati equation*) for B with subject to $B(0) = 0$. As shows up the solution of 1 does not depend on θ'. After that start with $A(0) = 0$ in formula 2.7 for solving 2:

$$\begin{aligned} A(t) &= \int_0^t A(s)\,ds \\ \dot{A} - B' \cdot \kappa \cdot \theta + \frac{1}{2} \cdot B' \cdot \Sigma \cdot \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \cdot \Sigma \cdot B &= \delta_0 \end{aligned}$$

where $\dot{A}$ is given be equating to δ_0.

2.9 Example: CIR model

This model is based on the process given in formula 2.1 on page 6:

$$dr(t) = \kappa \cdot (\theta - r(t))\, dt + \sigma \cdot \sqrt{r(t)} dW(t)$$

As this is a one factor model we have $Y = r$. Now we want to know the price $f(t, r)$ at time t of a bond maturing at u. First we try the affine form

$$f(t, r) = e^{-A(u-t)-B(u-t)\cdot r}$$

and set

$$Z = -A(u-t) - B(u-t) \cdot r$$

Then we calculate

$$\begin{aligned}
\frac{df}{f} &= dZ + \frac{1}{2}(dZ)^2 \\
&= \dot{A}dt + \dot{B} \cdot rdt - Bdr + \frac{1}{2} \cdot B^2 (dr)^2 \\
&= \dot{A}dt + \dot{B} \cdot rdt - \left(B \cdot \left[\kappa \cdot (\theta - r)\, dt + \sigma \cdot \sqrt{r} dW\right]\right) + \frac{1}{2} \cdot B^2 \cdot \sigma^2 \cdot rdt
\end{aligned}$$

Since $\frac{df}{f} = rdt +$ stochastic part we can find functions for A and B:

$$\dot{A} + \dot{B} \cdot r - B \cdot \kappa \cdot (\theta - r) + \frac{1}{2} \cdot B^2 \cdot \sigma^2 \cdot r = r$$

So the coefficient must match the *Riccati equation*

$$\dot{B} + B \cdot \kappa + \frac{1}{2} \cdot B^2 \cdot \sigma^2 = 1 \tag{2.8}$$

with the initial condition $B(0) = 0$. Further

$$\begin{aligned}
\dot{A} - B \cdot \kappa \cdot \theta &= 0 \qquad (2.9) \\
&\Downarrow \\
A(\tau) &= \kappa \cdot \theta \cdot \int_0^\tau B(s)\, ds
\end{aligned}$$

Now solve the Riccati equation 12 independent from θ, because θ only appears in formula 12. Suppose that θ is a function of t then set $\tau = u - t$ and $t = u - \tau$ so that $\theta = \theta(u - \tau)$ and finally

$$A(\tau) = \kappa \cdot \int_0^\tau B(s) \cdot \theta(u - s)\, ds$$

2.9.1 Fitting to current yield curve

The today's price of a bond maturing at u is

$$\underbrace{\hat{P}(0, u)}_{\text{market price}} = \underbrace{e^{-A(u)-B(u)\cdot r(0)}}_{\text{model price}}$$

We calculate

$$
\begin{aligned}
\log \hat{P}(0,u) &= -A(u) - B(u) \cdot r(0) \\
\hat{F}(0,u) &= -\frac{d \log \hat{P}(0,u)}{du} \\
&= \dot{A}(u) + \dot{B}(u) \cdot r(0) \\
&= \kappa \cdot B(u) \cdot \theta(u) + \dot{B}(u) \cdot r(0) \\
\hat{P}(0,u) &= e^{-\int_0^u \hat{F}(t) dt} \\
\dot{A}(u) &= \kappa \cdot B(u) \cdot \theta(u) \\
\theta(u) &= \frac{\hat{F}(0,u) - \dot{B}(u) \cdot r(0)}{\kappa \cdot B(u)}
\end{aligned}
$$

where F denotes the forward rate.

To fit the model to the current yield curve we have several possibilites:

1. Use time dependent parameters
2. Add a function of time
3. Model the forward rate (used by *Heath-Jarrow-Morton*)

2.9.2 Add independent factors

Suppose

$$r(t) = X_1(t) + X_2(t)$$

where X_1 and X_2 are independent stochastic processes under Q.

$$
\begin{aligned}
P(t,u) &= \mathbb{E}_t^Q \left[e^{-\int_t^u X_1(s) + X_2(s) ds} \right] \\
&= \mathbb{E}_t^Q \left[\underbrace{e^{-\int_t^u X_1(s) ds} \cdot e^{-\int_t^u X_2(s) ds}}_{\text{independent random variables}} \right] \\
&= \mathbb{E}_t^Q \left[e^{-\int_t^u X_1(s) ds} \right] \cdot \mathbb{E}_t^Q \left[e^{-\int_t^u X_2(s) ds} \right]
\end{aligned}
$$

Take an affine model with short rate r

$$P(t,u) = e^{-A(u-t) - B(u-t)' \cdot Y(t)}$$

and set

$$
\begin{aligned}
\hat{r}(0) &= r(0) \qquad \text{today's short rate} \\
\hat{r}(t) &= r(t) + X(t)
\end{aligned}
$$

for some deterministic function X starting at $X(0) = 0$. Discount bond prices are now

$$e^{-\int_0^u X(s)ds} \cdot e^{-A(u)-B(u)'\cdot Y(0)}$$

at time 0. Now match the yield curve

$$\underbrace{\log \hat{P}(0,u)}_{\text{market price}} = -\int_0^u X(s)\,ds - A(u) - B(u)' \cdot Y(0)$$

Take the derivative

$$\underbrace{\hat{F}(0,u)}_{-\frac{d\log P(0,u)}{du}} = X(u) + \dot{A}(u) + \dot{B}(u)' \cdot Y(0)$$

which tells us what X to choose to fit the yield curve:

$$X(u) = \hat{F}(0,u) - \dot{A}(u) - \dot{B}(u)' \cdot Y(0)$$

2.10 Example: Longstaff-Schwartz model

This model uses the following processes

$$\begin{aligned} r(t) &= Y_1(t) + Y_2(t) \\ dY_i &= \kappa_i \cdot (\theta_i - Y_i)\,dt + \sigma_i \cdot \sqrt{Y_i}dW_i \qquad \forall i = 1,2 \end{aligned}$$

for W_1 and W_2 are independent standard Brownian motions.

New factors:

$$\begin{aligned} Z_1 &= r \\ &= Y_1 + Y_2 \\ Z_2 &= \sigma_1^2 \cdot Y_1 + \sigma_2^2 \cdot Y_2 \\ Z &= \underbrace{\begin{pmatrix} 1 & 1 \\ \sigma_1^2 & \sigma_2^2 \end{pmatrix}}_{\text{constant}} \cdot Y \\ &= L \cdot Y \end{aligned}$$

So Z is a linear transformation of Y.

$$\begin{aligned} dZ &= LdY \\ &= L \cdot \kappa \cdot (\theta - Y)\, dt \\ dY &= \kappa \cdot (\theta - Y)\, dt + \Sigma \cdot \sqrt{S\,(t)\, dW} \\ \kappa &= \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \\ \theta &= \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\ \Sigma &= I \\ S &= \begin{pmatrix} \sigma_1^2 \cdot Y_1 & 0 \\ 0 & \sigma_2^2 \cdot Y_2 \end{pmatrix} \end{aligned}$$

Now check that Z is an affine model:

$$\begin{aligned} dY_i &= \sum_l \kappa_{il} \cdot (\theta_l - Y_l)\, dt + \sum_l \sigma_{il} \cdot \sqrt{\alpha_l + \beta_l' \cdot Y} dW_l \\ (dY_i) \cdot (dY_j) &= \sum_l \sigma_{il} \cdot \sigma_{jl} \cdot \left(\alpha_l + \beta_l' \cdot Y\right) dt \\ &= \sum_l \sigma_{il} \cdot \sigma_{jl} \cdot \alpha_l dt + \left(\sum_l \sigma_{il} \cdot \sigma_{jl} \cdot \beta_l\right)' \cdot Y dt \end{aligned}$$

We see that the drift and the covariances are affine functions of Y - this identifies an affine model. In the general model we have

$$\begin{aligned} (dY) \cdot (dY)' &= \Sigma \cdot S \cdot \Sigma' dt \\ \Sigma &= \text{identity matrix} \\ (dY) \cdot (dY)' &= \underbrace{S dt}_{\text{diagonal}} \end{aligned}$$

Further

$$\begin{aligned} Z &= L \cdot Y \\ dZ &= L \cdot dY \\ &= L \cdot \kappa \cdot (\theta - Y)\, dt + L \cdot \Sigma \cdot \sqrt{S} dW \\ &= \kappa^* \cdot (\theta^* - Z) + \Sigma^* \cdot \sqrt{S} dW \end{aligned}$$

If $\kappa^* \cdot \theta^* = L \cdot \kappa \cdot \theta$ and $\kappa^* \cdot Z = L \cdot \kappa \cdot Y$ or $\kappa^* \cdot L \cdot Y = L \cdot \kappa \cdot Y$ we have

$$\kappa^* = L \cdot \kappa \cdot L^{-1}$$

Further we calculate

$$\begin{aligned} \kappa^* \cdot \theta^* &= L \cdot \kappa \cdot \theta \\ L \cdot \kappa \cdot L^{-1} \cdot \theta^* &= L \cdot \kappa \cdot \theta \end{aligned}$$

So we get

$$\begin{aligned}
\theta^* &= L \cdot \theta \\
\Sigma^* &= L \cdot \Sigma \\
\beta_l^* &= \left(L^{-1}\right)' \cdot \beta_l \\
S &= \begin{pmatrix} \alpha_1 + \beta_1' \cdot Y & & 0 \\ & \ddots & \\ 0 & & \alpha_N + \beta_N' \cdot Y \end{pmatrix} \\
&= \begin{pmatrix} \alpha_1 + \beta_1' \cdot L^{-1} \cdot Z & & 0 \\ & \ddots & \\ 0 & & \alpha_N + \beta_N' \cdot L^{-1} \cdot Z \end{pmatrix} \\
&= \begin{pmatrix} \alpha_1 + \beta_1^{*'} \cdot Z & & 0 \\ & \ddots & \\ 0 & & \alpha_N + \beta_N^{*'} \cdot Z \end{pmatrix}
\end{aligned}$$

Our new factors are

$$\begin{aligned}
Z_1 &= Y_1 + Y_2 \\
Z_2 &= \sigma_1^2 \cdot Y_1 + \sigma_2^2 \cdot Y_2
\end{aligned}$$

where $Z_1 = r$ is the short rate and Z_2 is the variance of the short rate as we see here:

$$\begin{aligned}
(dr)^2 &= (dY_1)^2 + (dY_2)^2 + 2 \cdot (dY_1) \cdot (dY_2) \\
&= \left(\sigma_1^2 \cdot Y_1 + \sigma_2^2 \cdot Y_2\right) dt \\
&= Z_2 dt
\end{aligned}$$

Usually we call Z_2 v.

3 Brownian rotation

Let K be an orthogonal matrix, where the rows of K have unit length and are mutually orthogonal (i.e. orthogonal to itself: $K \cdot K' = I$). If W is a N-vector of independent standard Brownian motions, so

$$d\hat{W}(t) = K \cdot dW(t)$$

To check if it is a Brownian motion calculate

$$\hat{W}(t) = \int_0^t K(s)\, dW(s)$$

So $\hat{W}$ is a martingale and has unit variance

$$\begin{aligned} \left(d\hat{W}\right) \cdot \left(d\hat{W}\right)' &= K \cdot (dW) \cdot (dW)' \cdot K' \\ &= K \cdot I \cdot K' dt \\ &= I dt \end{aligned}$$

We calculate

$$\begin{aligned} K &= \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \\ K_{11} &= \sqrt{\frac{\sigma_1^2 \cdot Y_1}{\sigma_1^2 \cdot Y_1 + \sigma_2^2 \cdot Y_2}} \\ K_{12} &= \sqrt{\frac{\sigma_2^2 \cdot Y_2}{\sigma_1^2 \cdot Y_1 + \sigma_2^2 \cdot Y_2}} \\ K_{11}^2 + K_{12}^2 &= 1 \end{aligned}$$

Now take a look at the stochastic part of dr:

$$\begin{aligned} \text{stochastic part of } dr &= \sigma_1 \cdot \sqrt{Y_1} dW_1 + \sigma_2 \cdot \sqrt{Y_2} dW_2 \\ &= \sqrt{\sigma_1^2 \cdot Y_1 + \sigma_2^2 \cdot Y_2} \cdot (K_{11}, K_{12}) \cdot \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix} \\ &= \sqrt{v} d\hat{W}_1 \\ dv &= \sigma_1^2 dY_1 + \sigma_2^2 dY_2 \\ &= \sigma_1^2 \cdot \kappa_1 \cdot (\theta_1 - Y_1)\, dt + \sigma_2^2 \cdot \kappa_2 \cdot (\theta_2 - Y_2)\, dt + \\ &= +\sigma_1^3 \cdot \sqrt{Y_1} dW_1 + \sigma_2^3 \cdot Y_2 dW_2 \end{aligned}$$

In $d\hat{W} = KdW$ choose second row to be orthogonal to first and have unit length. In $dW = K'd\hat{W}$ look at the stochastic part

$$dv = \left(\sigma_1^3\sqrt{Y_1} + \sigma_2^3 \cdot \sqrt{Y_2}\right) \cdot K'd\hat{W}$$

If we go from the risk neutral measure to the actual measure, the drift parameters κ and θ will change.

4 Forward rate models

Define $F(t, s, u)$ as the continous compounded forward rate on loans from s to u that exists at time t with $t < s < u$. To create an investment at s, short sell one unit s-maturity bond and buy u-maturity bonds. So if we buy $\frac{P(t,s)}{P(t,u)}$ units of u-maturity bonds at time t we receive $\frac{P(t,s)}{P(t,u)}$ at time u. So we must have

$$\begin{aligned} \frac{P(t,s)}{P(t,u)} &= 1 + \text{rate of return} \\ &= e^{F(t,s,u)\cdot(u-s)} \end{aligned}$$

In terms of continous compounding this defines F as continously compounded rate. We view F as an annual rate.

$$\begin{aligned} \frac{\log P(t,s) - \log P(t,u)}{u-s} &= F(t,s,u) \\ \lim_{n\to s} \frac{\log P(t,s) - \log P(t,u)}{u-s} &= -\frac{d \log P(t,s)}{ds} \\ &= F(t,s) \end{aligned}$$

where $F(t, s)$ is the *forward rate* at time t for an instantaneous loan at $s > t$.

4.1 Vasicek model

This model defines

$$dr = \kappa \cdot (\theta - r)\, dt + \sigma dW$$

We start at time t and want to know the interest rate at a future time point $u > t$

$$r(u) = r(t) + \left(1 + e^{-\kappa\cdot(u-t)}\right) \cdot (\theta - r(t)) + \sigma \cdot \int_t^u e^{-\kappa(u-s)} dW(s)$$

where $r(u)$ is normally distributed with mean

$$r(t) + \left(1 - e^{-\kappa\cdot(u-t)}\right) \cdot (\theta - r(t))$$

and variance

$$\sigma^2 \cdot \int_t^u e^{-2\cdot\kappa\cdot(u-2)} ds$$

If we look ath this model without drift, i.e. with no mean-reversion, we have for $u > t$

$$\begin{aligned} dr &= \sigma dW \\ r(u) &= r(t) + \sigma \int_t^u dW(s) \\ P(t,u) &= \mathbb{E}^Q \left[e^{-\int_t^u r(s)ds} \middle| r(t) \right] \\ \int_t^u r(s)\,ds &= \int_t^u \left[r(t) + \sigma \cdot \int_t^s dW(a) \right] ds \\ &= (u-t) \cdot r(t) + \sigma \cdot \int_t^u \int_t^s dW(a)\,ds \end{aligned}$$

Now we change the order of integration an get

$$\begin{aligned} \int_t^u r(s)\,ds &= (u-t) \cdot r(t) + \sigma \int_t^u \int_a^u ds dW(a) \\ &= (u-t) \cdot r(t) + \sigma \int_t^u (u-a)\,dW(a) \end{aligned}$$

The process $\int_t^u r(s)\,ds$ is normally distributed with mean

$$(u-t) \cdot r(t)$$

and variance

$$\sigma^2 \cdot \int_t^u (u-a)^2\,da = \sigma^2 \cdot \frac{(u-t)^3}{3}$$

Finally we get

$$P(t,u) = e^{-(u-t)\cdot r(t) + \frac{\sigma^2}{6}(u-t)^3}$$

as the bond pricing formula for the non-mean reverting Vasicek model.

Now consider the forward price

$$\begin{aligned} \log P(t,u) &= -(u-t)\,r(t) + \frac{\sigma^2}{6} \cdot (u-t)^3 \\ F(t,u) &= -\frac{d \log P(t,u)}{du} \\ &= r(t) - \frac{\sigma^2}{2} \cdot (u-t)^2 \end{aligned}$$

If we want to know how the forward rate changes over time we have to fix the maturity date u and take differential with respect to t

$$\begin{aligned} dF(t,u) &= dr + \sigma^2 \cdot (u-t)\,dt \\ &= (u-t) \cdot \sigma^2 dt + \sigma dW \end{aligned}$$

4.2 Ho-Lee Model

This model uses for r_i a continously compounded annualized rate at time t_i for loans from t_i to t_{i+1}, where $\Delta t = t_i - t_{i-1}$ is fixed. The discount bond price at t_i maturing next day is

$$P\left(t, t_{i+1}\right) = e^{-r_i \cdot \Delta t}$$

For a fixed annual variance σ and two constant parameters θ_1 and θ_2 we have

$$\begin{array}{ccccc} & & & & r_0+\theta_1+\theta_2+2\cdot\sigma\cdot\sqrt{\Delta t} \\ & & & \nearrow & \\ & & r_0+\theta_1+\sigma\cdot\sqrt{\Delta t} & & \\ & \nearrow & & \searrow & \\ r_0 & & & & r_0+\theta_1+\theta_2 \\ & \searrow & & \nearrow & \\ & & r_0+\theta_1-\sigma\cdot\sqrt{\Delta t} & & \\ & & & \searrow & \\ & & & & r_0+\theta_1+\theta_2-2\cdot\sigma\cdot\sqrt{\Delta t} \end{array}$$

Under Q the probabilities are $\frac{1}{2}$.

$$\begin{aligned} r_1 &= r_0 + \theta_1 + Z_1 \\ r_2 &= r_1 + \theta_2 + Z_2 \\ Z_1 &= \pm\sigma \cdot \sqrt{\Delta t} \\ Z_2 &= \pm\sigma \cdot \sqrt{\Delta t} \end{aligned}$$

For two dates $t_i < t_k$ we have

$$\begin{aligned} P\left(t_i, t_k\right) &= \mathbb{E}^Q\left[e^{-\sum_{l=i}^{k-1} r_l \cdot \Delta t} \middle| r_i \right] \\ r_l &= r_i + \sum_{h=i+1}^{l} \left(\theta_h + Z_h\right) \qquad \forall l \geq i+1 \\ \sum_{l=1}^{k-1} r_l &= r_i + \sum_{l=i+1}^{k-1} r_l \\ &= r_i + \sum_{l=i+1}^{k-1} \left(r_i + \sum_{h=i+1}^{l} \left(\theta_h + Z_h\right) \right) \\ &= (k-i) \cdot r_i + \sum_{l=i+1}^{k-1} \sum_{h=i+1}^{l} \left(\theta_h + Z_h\right) \end{aligned}$$

where only Z is random. Further

$$P\left(t_i, t_k\right) = e^{-(k-i) \cdot r_i \cdot \Delta t - \sum_{l=i+1}^{k-1} \sum_{h=i+1}^{l} \theta_n \cdot \Delta t} \cdot \underbrace{\mathbb{E}^Q\left[e^{-\sum_{l=i+1}^{k-1} \sum_{h=i+1}^{l} Z_n \cdot \Delta t} \right]}_{e^{\alpha \cdot (k-i-1)}}$$

For two periods we can set

$$\begin{aligned}\mathbb{E}^Q\left[e^{-Z_1\cdot\Delta t}\right] &= e^{\alpha(1)}\\ &= \frac{1}{2}\cdot e^{-\left(\sigma\cdot\sqrt{\Delta t}\right)\cdot\Delta t}+\frac{1}{2}\cdot e^{\left(\sigma\cdot\sqrt{\Delta t}\right)\cdot\Delta t}\end{aligned}$$

and then we get for $P(t_i, t_k)$

$$P(t_i,t_k) = e^{-(k-i)\cdot r_i\cdot\Delta t-\sum_{l=i+1}^{k-1}\sum_{h=i+1}^{l}\theta_n\cdot\Delta t}+\alpha\cdot(k-i-1)$$

As bond pricing formula for time 0 we have

$$P(0,t_n) = e^{-n\cdot r_0\cdot\Delta t-\sum_{l=1}^{n-1}\sum_{h=1}^{l}\theta_h\cdot\Delta t+\alpha(n-1)}$$

and further

$$\begin{aligned}P(0,t_{n+1}) &= e^{-(n+1)\cdot r_0\cdot\Delta t-\sum_{l=1}^{n}\sum_{h=1}^{l}\theta_h\cdot\Delta t+\alpha(n)}\\ \frac{P(0,t_n)}{P(0,t_{n+1})} &= e^{r_0\cdot\Delta t+\sum_{h=1}^{n}\cdot\Delta t+\alpha(u-1)-\alpha(u)}\\ P(0,t_2) &= e^{-2\cdot r_0\cdot\Delta t-\theta_1\cdot\Delta t+\alpha(1)}\\ P(0,t_3) &= e^{-3\cdot r_0\cdot\Delta t-2\cdot\theta_1\cdot\Delta t-\theta_2\cdot\Delta t+\alpha(2)}\\ P(0,t_4) &= e^{-4\cdot r_0\cdot\Delta t-3\cdot\theta_1\cdot\Delta t-2\cdot\theta_2\cdot\Delta t-\theta_3\cdot\Delta t+\alpha(3)}\end{aligned}$$

We have to choose θ_1 to fit the market price of a two period bond and θ_2 to fit the market price of a three period bond.

$$\begin{aligned}r_{t_i} &= r_{t_{i-1}}+\theta_i+Z_i\\ \Delta r_i &= \theta_i+Z_i\\ dr(t) &= \theta(t)\,dt+\sigma dW\end{aligned}$$

where θ is a deterministic function. Remember that the forward rate $F(0,t_n,t_{n+1})$ is defined by

$$e^{F(0,t_n,t_{n+1})\cdot\Delta t} = 1+\text{rate of return}$$

Definition: The forward rate is defined by

$$\begin{aligned}F(0,t_n,t_{n+1}) &= e^{F(0,t_n,t_{n+1})\cdot\Delta t}\\ &= \frac{P(0,t_n)}{P(0,t_{n+1})}\\ F(0,t_n,t_{n+1}) &= r_0+\sum_{h=1}^{n}\theta_h+\frac{\alpha(n-1)-\alpha(n)}{\Delta t}\\ F(1,t_n,t_{n+1}) &= r_1+\sum_{h=2}^{n}\theta_n+\frac{\alpha(n-2)-\alpha(n-1)}{\Delta t}\\ P(1,t_n) &= e^{-(n-1)\cdot r_1\cdot\Delta t-\sum_{l=2}^{n-1}\sum_{h=2}^{l}\theta_n\cdot\Delta t+\alpha(n-2)}\\ P(1,t_{n+1}) &= e^{-n\cdot r_1\cdot\Delta t-\sum_{l=2}^{n}\sum_{h=2}^{l}\theta_n\cdot\Delta t+\alpha(n-1)}\\ \frac{P(1,t_n)}{P(1,t_{n+1})} &= e^{-r_1\cdot\Delta t+\sum_{h=2}^{n}\theta_n\cdot\Delta t+\alpha(n-2)-\alpha(n-1)}\end{aligned}$$

The change of the forward rate is equal to

$$\begin{aligned}
\Delta F &= F\left(1, t_u, t_{u+1}\right) - F\left(0, t_u, t_{u+1}\right) \\
&= r_1 - r_0 - \theta_1 + \frac{\alpha\left(n-2\right) - 2\cdot\alpha\cdot\left(n-1\right) + \alpha\left(n\right)}{\Delta t} \\
&= Z_1 + \frac{\alpha\left(n-2\right) - 2\cdot\alpha\left(n-1\right) + \alpha\left(1\right)}{\Delta t}
\end{aligned}$$

where Z_1 is independent of the maturity and the second term depends on the maturity. So we can write

$$dF\left(t,u\right) = \beta\left(u-t\right)dt + \sigma dW$$

Once again β is a function of the maturity and σdW is a random price. This reminds us of Vasicek with $\kappa = 0$ with no mean reversion:

$$dF\left(t,u\right) = \underbrace{\left(u-t\right)\cdot\sigma^2}_{\text{equiv. to }\beta}dt + \sigma dW$$

5 Heath-Jarrow-Morton models

Heath-Jarrow-Morton is a type of writing known models; it fixes date u and consider $F(t,u)$ under Q as t increases. Assume

$$dF(t,u) = \mu(t,u)\,dt + \sum_{i=1}^{n} \sigma_i(t,u)\,dW_i(t)$$

where $\mu(t,u)$ and $\sigma_i(t,u)$ are at time t know stochastic processes which may depend on the history of t. The initial condition is that $F(0,u)$ is given as the market forward rate at time 0 for every u. Also given is

$$P(0,u) = e^{-\int_0^u F(0,s)ds}$$

which matches the current yield curve.

The *result of HJM* is that the σ_i's determine the μ's. Take $N = 1$:

$$\begin{aligned} dF(t,u) &= \mu(t,u)\,dt + \sigma(t,u)\,dW(t) \\ \frac{dP(t,u)}{P(t,u)} &= r(t)\,dt + \text{stochastic part} \\ P(t,u) &= e^{-\int_t^u F(t,s)ds} \end{aligned}$$

Now set

$$Y(t) = \int_t^u F(t,s)\,ds$$

and with usual calculus we get

$$\begin{aligned} \frac{d}{dt}\int_t^u F(t,s)\,ds &= -F(t,t) + \int_t^u \frac{\partial F(t,s)}{\partial t} ds \\ dY &= -F(t,t)\,dt + \int_t^u (dF(t,s))\,ds \\ F(t,t) &= r(t) \\ dY(t) &= -r(t)\,dt + \left\{\int_t^u (\mu(t,s)\,ds)\right\} dt + \left\{\int_t^u (\sigma(t,s))\,ds\right\} dW(t) \\ (dY)^2 &= \left\{\int_t^u \sigma(t,s)\,ds\right\}^2 dt \end{aligned}$$

Now write

$$f(u) = \int_t^u \sigma(t,s)\,ds$$

and calculate

$$\begin{aligned}
(dY)^2 &= f(u)^2\,dt \\
&= f(t)^2 + 2\cdot\int_t^u f(s)\cdot f'(s)\,ds \\
&= 2\cdot\int_t^u \sigma(t,s)\cdot\int_t^s \sigma(t,a)\,dads \\
\frac{dP}{P} &= -dY + \frac{1}{2}\cdot(dY)^2 \\
&= rdt - \left\{\int_t^u \mu(t,s)\,ds\right\}dt + \underbrace{\text{stochastic part}}_{\left\{\int_t^u \sigma(t,a)da\right\}dW(t)} + \left\{\int_t^u \sigma(t,s)\cdot\int_t^s \sigma(t,a)\,dads\right\}dt \\
&= rdt + \text{stochastic part}
\end{aligned}$$

This now leads us to the *HJM result*:

$$\begin{aligned}
\int_t^u \mu(t,s)\,ds &= \int_t^u \sigma(t,s)\cdot\int_t^s \sigma(t,a)\,dads \qquad \forall u \\
\mu(t,s) &= \sigma(t,s)\cdot\int_t^s \sigma(t,a)\,da
\end{aligned}$$

where $\sigma(t,s)$ is the volatility coefficient of $dF(t,s)$ and $\int_t^s \sigma(t,a)\,da$ is the volatility coefficient of the discount bond $\frac{dP(t,s)}{P(t,s)}$.

As final result we get

$$\begin{aligned}
dF(t,u) &= \sigma(t,u)\cdot\left\{\int_t^u \sigma(t,a)\,da\right\}dt + \sigma(t,u)\,dW(t) \\
\frac{dP(t,u)}{P(t,u)} &= rd - \left\{\int_t^u \sigma(t,a)\,da\right\}dW(t)
\end{aligned}$$

A special case would be $\sigma(t,u) = \sigma$ with gives us

$$dF(t,u) = (u-t)\cdot\sigma^2 dt + \sigma dW(t)$$

which is a Vasicek model with $\kappa = 0$.

5.1 Derivatives

Consider a call option maturing at T on a discount bond maturing at $u > T$ with exercise price K. The price of that option at time 0 is

$$\begin{aligned}
&\mathbb{E}^Q\left[e^{-\int_0^T r(s)ds}\cdot(P(T,u)-K)^+\right] = \\
&= \underbrace{\mathbb{E}^Q\left[e^{-\int_0^T r(s)ds}\cdot P(T,u)\cdot I_{P(T,u)>K}\right]}_{\diamondsuit} - \underbrace{K\cdot\mathbb{E}^Q\left[e^{-\int_0^T r(s)ds}\right]}_{\clubsuit}
\end{aligned} \tag{5.1}$$

where $\diamondsuit$ use a bond maturing at u as numeraire and $\clubsuit$ uses a bond maturing at T as numeraire.

To solve $\diamondsuit$ using ρ as the state prices calculate

$$\begin{aligned}
\frac{dQ}{dP} &= e^{-\int_0^T r(s)ds} \cdot \rho(T) \\
\frac{dQ^*}{dP} &= \frac{P(T,u)}{P(0,u)} \cdot \rho(T) \\
\frac{dQ^*}{dQ} &= \frac{dQ^*}{dP} \cdot \frac{dP}{dQ} \\
&= \frac{P(T,u)}{P(0,u)} \cdot e^{-\int_0^T r(s)ds} \\
\mathbb{E}^Q\left[e^{-\int_0^T r(s)ds} \cdot P(T,u) \cdot I_{P(T,u)>K}\right] &= P(0,u) \cdot \mathbb{E}^Q\left[\frac{dQ^*}{dQ} \cdot I_{P(T,u)>K}\right] \\
&= P(0,u) \cdot Q^*\left(P(T,u) > K\right)
\end{aligned}$$

If we use the same procedure to solve $\clubsuit$ we get

$$K \cdot \mathbb{E}^Q\left[e^{-\int_0^T r(s)ds}\right] = K \cdot P(0,T) \cdot Q^{**}\left(P(T,u) > K\right)$$

where Q^{**} uses the T-maturity bond as numeraire.

To actually calculate a value for the Q^* respective Q^{**} probability we need

Girsonov's Theorem: If W is as Q-Brownian motion, then $dW^* = dW - \frac{d\xi}{\xi}dW$ defines a Q^*-Brownian motion, where we first start under Q and $\frac{d\xi}{\xi}$ is equal to the stochastic part of the new numeraire's return.

In this case assume a Vasicek model with $\kappa = 0$ which is equivalent to a constant volatility HJM.

$$\begin{aligned}
\frac{dP(t,u)}{P(t,u)} &= rdt - \left(\int_t^u \sigma(t,s)\,ds\right) dW(t) \\
&= rdt - (u-t) \cdot \sigma dW(T)
\end{aligned}$$

Now set $dW^* = dW + (u-t) \cdot \sigma dW \cdot dW = dW + (u-t) \cdot \sigma dt$ and plug in

$$\begin{aligned}
dP(t,u) &= rdt - (u-t) \cdot \sigma \cdot \{dW^* - (u-t) \cdot \sigma dt\} \\
&= r + \left((u-t)^2 \cdot \sigma^2\right) dt - (u-t) \cdot \sigma dW^* \\
P(T,u) &= P(0,u) \cdot e^{\int_0^T \left(r(t) + (u-t)^2 \cdot \sigma^2\right) dt - \int_0^T (u-t) \cdot \sigma dW^*(t) - \frac{1}{2} \cdot \int_0^T (u-t)^2 \cdot \sigma^2 dt}
\end{aligned}$$

The options is in the money iff

$$\log P(0,u) + \int_0^T r(t)\,dt + \int_0^T (u-t)^2 \cdot \sigma^2 dt - \int_0^T (u-t) \cdot \sigma dW^*(t) - \frac{1}{2} \cdot \int_0^T (u-t)^2 \cdot \sigma^2 dt > \log K$$

The model for r is

$$r(t) = r(0) + \theta(t) + \sigma \cdot W(t)$$

for some θ, which is a deterministic function of time t. As final result we have

$$Q^*(P(t,u) > K) = N[d_1]$$

for some d_1.

We are using the same method to solve ♣ of formula 5.1 while now using a T-maturity bond as measure which gives us

$$\frac{d\xi}{\xi} = -(T-t) \cdot \sigma dW$$

Vasicek model with $\kappa = 0$: For a Vasicek model with $\kappa = 0$ we get a constant volatility HJM-model. For a constant σ we have under Q

$$\begin{aligned} \frac{dP(t,u)}{P(t,u)} &= rdt + \int_t^u \sigma ds dW(t) \\ &= rdt + (u-t) \cdot \sigma dW(t) \end{aligned}$$

6 Swaps

Assume a two year fixed for floating interest rate swap with six month payments and a notional principle of one dollar. The fixed rate payer (i.e. the long position of the swap) has to pay every six months $\frac{R}{2}$ times the notional principle to the floating rate payer, where R is the fixed rate. The floating rate payer has to pay every six months $\frac{L}{2}$ times the notional principle to the fixed rate payer, where L denotes the Libor rate at the beginning of the last six months.

A replicating portfolio would be:

- long one 6-month bond, short $1 + \frac{R}{2}$ 1-year bonds
- long one 1-year bond, short $1 + \frac{R}{2}$ 1.5-years bonds
- long one 1.5-year bond, short $1 + \frac{R}{2}$ 2-years bonds

At $t = 0.5$ we receive one dollar from the 6-month bond and invest that for six months at the LIBOR. In one year we have $1 + \frac{L}{2}$ from the LIBOR investment and we owe $1 + \frac{R}{2}$ from the short position in the 1 year bonds. This gives us a net value of

$$\frac{L}{2} - \frac{R}{2}$$

At time 0 the portfolio must have a value of zero, i.e. the fair swap rate R must satisfy

$$P\left(0, 0.5\right) - \frac{R}{2} \cdot P\left(0, 1\right) - \frac{R}{2} \cdot P\left(0, 1.5\right) - \left(1 + \frac{R}{2}\right) \cdot P\left(0, 2\right) = 0$$

7 Swaptions

As an example, a swaption gives you the right in one year to enter into a two year swap as the fixed rate payer with fixed rate R (i.e. sell the swaption; put swaption; floating rate payer buys the swap). What is it worth?

The option will be exercised if

$$P(1,1.5) - \frac{R}{2} \cdot P(1,2) - \frac{R}{2} \cdot P(1,2.5) - \left(1 + \frac{R}{2}\right) \cdot P(1,3) > 0$$

The value of option at time 0 is

$$\mathbb{E}^Q \left[e^{-\int_0^1 r(t)dt} \cdot \left\{ P(1,1.5) - \frac{R}{2} \cdot P(1,2) - \frac{R}{2} \cdot P(1,2.5) - \left(1 + \frac{R}{2}\right) \cdot P(1,3) \right\}^+ \right]$$

If there is a payment on the swap at six months, the fixed rate payer's equivalent portfolio is worth $\frac{L(1)}{2} - \frac{R}{2}$, where $L(1)$ denotes the six-month LIBOR at time 1. The value of the swaption at time 0 is

$$\mathbb{E}^Q \left[e^{-\int_0^1 r(t)dt} \cdot \left\{ \left(\frac{L(1) - R}{2} + 1 \right) \cdot P(1,1.5) - \frac{R}{2} \cdot P(1,2) - \frac{R}{2} \cdot P(1,2.5) - \left(1 + \frac{R}{2}\right) \cdot P(1,3) \right\}^+ \right]$$

At time 0 the six-month LIBOR satisfies

$$\frac{1}{1 + \frac{L}{2}} = P(0,0.5) = e^{-\int_0^{0.5} F(0,t)dt}$$

8 Cap

The owner of a two-year cap with cap rate R receives each six months $\left(\frac{L}{2}-\frac{R}{2}\right)^{+}$. A Cap is equivalent to a portfolio of caplets.

Let $L\left(t,T,T+\delta\right)$ denote the forward rate at t for loans between T and $T+\delta$, expressed as annual rate, and *not* continously compounded. Let $L\left(T,T,T+\delta\right)$ denote the spot LIBOR at time T for δ-lenght loans. Then we can write

$$\frac{1}{1+\delta\cdot L\left(T,T,T+\delta\right)}=P\left(T,T+\delta\right)$$

The value of a single caplet would be

$$\mathbb{E}^{Q}\left[e^{-\int_0^1 r(t)dt}\cdot\left\{\delta\cdot L\left(1,1,1+\delta\right)-\delta R\right\}^{+}\right]$$

at time 1.

If we assume a Vasicek model with no mean reversion we get

$$\begin{aligned}1+\delta\cdot L\left(T,T,T+\delta\right) &= e^{\int_T^{T+\delta}F(T,s)ds}\\ 1+\delta\cdot L\left(t,T,T+\delta\right) &= e^{\int_T^{T+\delta}F(t,s)ds}\end{aligned}$$

Now set $\sigma\left(t,s\right)=\sigma$:

$$\begin{aligned}dF\left(t,s\right) &= \left(s-t\right)\cdot\sigma^2 dt-\left(s-t\right)\cdot\sigma dW\left(t\right)\\ dY &= \int_T^{T+\delta}dF\left(t,s\right)ds\\ &= \left(\int_T^{T+\delta}\left(s-t\right)\cdot\sigma^2 ds\right)dt-\left(\int_T^{T+\delta}\left(s-t\right)\cdot\sigma ds\right)dW\left(t\right)\\ &= \frac{d\left(1+\delta L\right)}{1+\delta L}\\ \frac{\delta dL\left(t,T,T+\delta\right)}{1+\delta L\left(t,T,T+\delta\right)} &= dY+\frac{1}{2}\left(dY\right)^2\\ &= \underbrace{0dt}_{\text{martingale}}+\delta\cdot\sigma dW^{*}\end{aligned}$$

Remember HJM constant volatility:

$$\begin{aligned} F(t,u) &= \left(\sigma(t,u)\cdot\int_t^u \sigma(t,s)\,ds\right)dt + \sigma(t,u)\,dW(t) \\ &= (u-t)\cdot\sigma^2 dt + \sigma dW(t) \\ \frac{dP(t,u)}{P(t,u)} &= rdt - \left(\int_t^u \sigma(t,s)\,ds\right)dW \\ &= rdt - (u-t)\cdot\sigma dW(t) \end{aligned}$$

If we use $Y = \int_T^{T+\delta} F(t,s)\,ds$ we get

$$\begin{aligned} dY &= \left(\int_T^{T+\delta}(s-t)\cdot\sigma^2 ds\right)dt + \left(\int_T^{T+\delta}\sigma ds\right)dW(t) \\ &= \left[\frac{(T+\delta-t)^2}{2} - \frac{(T-t)^2}{2}\right]\cdot\sigma^2 dt + \delta\cdot\sigma dW(t) \\ (dY)^2 &= \delta^2\cdot\sigma^2 dt \end{aligned}$$

Further

$$\begin{aligned} \frac{d\xi}{\xi} &= -\left(\int_T^{T+\delta}\sigma(t,s)\,ds\right)dW(t) \\ &= -(T+\delta-t)\cdot\sigma dW(T) \\ dW^* &= dW + \sigma\left(\dot{T}+\delta-t\right)dt \end{aligned}$$

With dW^* as Q^*-Brownian motion we get

$$Y = \left[\frac{(T+\delta-t)^2}{2} - \frac{(T-t)^2}{2}\right]\cdot\sigma^2 dt + \frac{1}{2}\cdot\delta^2\cdot\sigma^2 dt + \delta\cdot\sigma dW^*(t) - \delta\cdot\sigma^2\cdot(T+\delta-t)\,dt$$

The price of a caplet is therefore given by

$$\begin{aligned} \mathbb{E}^Q\left[e^{-\int_0^{T+\delta} r(t)dt}\cdot\{L(T,T,T+\delta)\cdot\delta - \delta\cdot R\}^+\right] &= P(0,T+\delta)\cdot\mathbb{E}^{Q^*}\left[(\delta\cdot L(T,T,T+\delta) - \delta\cdot R)^+\right] \\ &= P(0,T+\delta)\cdot\mathbb{E}^{Q^*}\left[(Z(T) - 1 - \delta\cdot R)^+\right] \\ &= P(0,T+\delta)\cdot BSC \end{aligned}$$

where $Z(T) = 1 + \delta\cdot L(T,T,T+\delta)$ and BSC denotes the Black-Scholes price for a call on a stock with volatility $\delta\cdot\sigma$, exercise price $1+\delta R$, initial stock price $1+\delta\cdot L(0,T,T+\delta)$ and interest rate zero ($r=0$).

Further we can calculate

$$\begin{aligned} \delta dL(t,T,T+\delta) &= (1+\delta\cdot L(t,T,T+\delta))\cdot\delta\cdot\sigma dW^*(t) \\ \frac{dL(t,T,T+\delta)}{L(t,T,T+\delta)} &= \underbrace{\frac{1+\delta\cdot L(t,T,T+\delta)}{\delta\cdot L(t,T,T+\delta)}}_{\text{BGM model assume constant}}\cdot\delta\cdot\sigma dW^*(t) \end{aligned}$$

The price of a caplet is then

$$P(0, T+\delta) \cdot \mathbb{E}^{Q^*}\left[\left(L(T, T, T+\delta) \cdot \delta - \delta \cdot R\right)^+\right] = P(0, T+\delta) \cdot \delta \cdot BSC$$

where BSC here is the Black-Scholes price for a call on a stock with constant volatility, exercise price R, initial stock price $L(0, T, T+\delta)$ and interest rate zero ($r = 0$).

9 Duffie-Singleton

Define $\Lambda(t)$ as following

$$\Lambda(t) = \begin{cases} 0 & \text{if no default before } t \text{ ; also if bond pays immediately} \\ 1 & \text{if default before } t \end{cases}$$

Also define the following:

- Let $P(t)$ be the price of the bond
- Let $\hat{P}(t)$ be the price assuming no default, $\hat{P}(T) = X$
- Let $\hat{P}(t-)$ be the price just before t: $\hat{P}(t-) = \lim_{u \to t} \hat{P}(u) \quad \forall u < t$
- Let $L(t)$ be the loss in market value at default. So $1-L(t)$ is the recovery and $(1 - L(t)) \cdot \hat{P}(t-)$ is the cash payout in the event of default, i.e. how much to recover if default.
- Let $h(t)\,dt$ be the probability of default in the instant dt at time t.
- Know that $\Lambda(t) - \int_0^t h(s)\,ds$ is a Q-martingale. This implies $\mathbb{E}[d\Lambda] - h(t)\,dt = 0$.
- We call h the *compensator* of the point process Λ.
- Let T be the maturity date.
- Let X be the payout at maturity if no default. So in case of a bond $X = 1$; otherwise X might take any value.

9.1 Gains process

The gains process is defined by

$$G(t) = (1 - \Lambda(t)) \cdot \hat{P}(t) + \int_0^t e^{\int_s^t r(u)du} \cdot (1 - L(t)) \cdot \hat{P}(s)\,d\Lambda(s)$$

where s would be the default date.

If $\Lambda(t) = 0$ we have $G(t) = \hat{P}(t)$, otherwise we have $G(t) = e^{\int_s^t r(u)du} \cdot (1 - L(t)) \cdot \hat{P}(s)$ in case $d\Lambda(s) = 1$. We can calculate

$$\begin{aligned} dG(t) &= (1+\Lambda)\, d\hat{P} - \hat{P} d\Lambda + (1 - L(t)) \cdot \hat{P}(t)\, d\Lambda(t) \\ &= r \cdot \hat{P} dt + \text{martingale part} \end{aligned}$$

If we assume no default before t then we have

$$\begin{aligned} G(t-) &= \hat{P}(t-) \\ \frac{dG(t)}{G(t-)} &= r dt + \underbrace{dM}_{\text{stoch. part}} \\ dG(t) &= r \cdot \hat{P}(t-)\, dt + \hat{P}(t-)\, dM \end{aligned}$$

where M is as Q-martingale and so $\int_0^t G dM$ is also a Q-martingale.

Now take $\frac{d\hat{P}}{\hat{P}} = \mu dt +$ martingale part, so that the expected return of $\hat{P}$ is μ; but as we can't buy $\hat{P}$ on the market, what is μ?

$$\begin{aligned} dG &= (1-\Lambda) \cdot \left\{ \mu \cdot \hat{P} dt + \text{another martingale} \right\} - L(t) \cdot \hat{P} d\Lambda \\ \frac{dG}{d\hat{P}} &= (1-\Lambda) \cdot \mu dt + \text{martingale} - L d\Lambda \\ &= (1-\Lambda) \cdot \mu dt - L \cdot \underbrace{(d\Lambda - h dt)}_{\text{martingale}} - h \cdot L dt + \text{martingale} \\ &= \underbrace{\{(1-\Lambda) \cdot \mu - h \cdot L\}}_{\text{interest rate}} dt + \text{martingale} \end{aligned}$$

where $\frac{dG}{d\hat{P}}$ is the return from holding the bond. It follows that

$$(1-\Lambda) \cdot \mu - h \cdot L = r$$

and if we assume no default

$$\mu = r + h \cdot L$$

so the fictive asset (bond) has a higher return than r - this will never default. We can further calculate

$$\begin{aligned} e^{-\int_0^t [r(s)+h(s)\cdot L(s)]ds} \cdot \hat{P}(t) &= \mathbb{E}^Q \left[e^{-\int_0^T (r(s)+h(s)\cdot L(s))ds} \cdot \hat{P}(T) \middle| \mathcal{F}_t \right] \\ \hat{P}(t) &= \mathbb{E}^Q \left[e^{-\int_t^T [r(s)+h(s)\cdot L(s)]ds} \cdot X \middle| \mathcal{F}_t \right] \end{aligned}$$

because of $\hat{P}(T) = X$.

Example

Assume the following:

$$\begin{aligned} dr &= \kappa \cdot (\theta - r)\, dt + \sigma \cdot \sqrt{r} dW_1 \\ y &= h \cdot L \\ dy &= \gamma \cdot (\phi - y)\, dt + \lambda \cdot \sqrt{y} dW_2 \end{aligned}$$

where W_1 and W_2 are independent as well as both square-root processes. Before the default we have

$$P(t) = \mathbb{E}^Q \left[e^{-\int_t^T r(s)ds} \cdot e^{-\int_t^T y(s)ds} \cdot X \middle| \mathcal{F}_t \right]$$

Take X as discount bond, i.e. $X = 1$.

$$\begin{aligned} P(t) &= \mathbb{E}^Q \left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] \cdot \mathbb{E}^Q \left[e^{-\int_t^T y(s)ds} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^Q \left[e^{-\int_t^T r(s)ds} \middle| r(t) \right] \cdot \mathbb{E}^Q \left[e^{-\int_t^T y(s)ds} \middle| y(t) \right] \\ &= \text{product of CIR} - \text{model square root bond prices} \end{aligned}$$

10 Ahn-Dittmar-Gallant (no default risk)

We work under the actual probability measure P. The price of a T-maturity discount bond is equal to

$$\mathbb{E}\left[\rho\left(T\right)\right] = \mathbb{E}^{Q}\left[e^{-\int_0^T r(t)dt}\right]$$

where ρ is a stochastic discount factor, i.e. a state price density process.

$$\begin{aligned}
\frac{dQ}{dP} &= e^{\int_0^T r(t)dt} \cdot \rho\left(T\right) \\
&= \xi\left(T\right) \\
\xi\left(t\right) &= \mathbb{E}\left[\left.\xi\left(T\right)\right| \mathcal{F}_t\right] \\
&= Q - \text{martingale}
\end{aligned}$$

Now define $\rho\left(t\right) = e^{-\int_0^t r(s)ds} \cdot \xi\left(t\right)$ and consider a security that pays X at time $u < T$, which price must be

$$\begin{aligned}
\mathbb{E}^{Q}\left[e^{-\int_0^u r(t)dt} \cdot X\right] &= \mathbb{E}\left[\xi\left(T\right) \cdot e^{-\int_0^u r(t)dt} \cdot X\right] \\
&= \mathbb{E}\left[\rho\left(u\right) \cdot X\right] \\
\mathbb{E}\left[\left.\xi\left(T\right) \cdot e^{-\int_0^u r(t)dt}\right| \mathcal{F}_u\right] &= e^{-\int_t^u r(t)dt} \cdot X \cdot \mathbb{E}\left[\left.\xi\left(T\right)\right| \mathcal{F}_u\right] \\
&= e^{-\int_0^u r(t)dt} \cdot X \cdot \xi\left(u\right) \\
&= \rho\left(\right) \cdot X
\end{aligned}$$

Remember that at time u we know ρ and the payoff. Using the law of iterated expectations the price of the security is

$$\mathbb{E}\left[\mathbb{E}\left[\left.\xi\left(T\right) \cdot e^{-\int_0^u r(t)dt} \cdot X\right| \mathcal{F}_u\right]\right] = \mathbb{E}\left[\rho\left(u\right) \cdot X\right]$$

Key result: For any reinvested asset price process S the term

$$\rho\left(t\right) \cdot S\left(t\right)$$

is a P-martingale and the term

$$e^{-\int_0^t r(u)du} \cdot S\left(t\right)$$

is a Q-martingale.

The fact that $\rho(t) \cdot S(t)$ is a P-martingale implies for all $t < u$

$$\rho(t) = \mathbb{E}\left[\rho(u) \cdot S(u) \middle| \mathcal{F}_t\right]$$

and so we get as pricing formula under P

$$S(t) = \mathbb{E}\left[\frac{\rho(u)}{\rho(t)} \cdot S(u) \middle| \mathcal{F}_t\right]$$

where the term $\frac{\rho(u)}{\rho(t)}$ is known as the *marginal rate of substituion.* Under Q we can write the pricing formula as

$$S(t) = \mathbb{E}^Q\left[e^{-\int_t^u r(a)da} \cdot S(u) \middle| \mathcal{F}_t\right]$$

Proof: To proof that $\rho(t) \cdot S(t)$ is a P-martingale we have to show that

$$\mathbb{E}\left[\rho(t) \cdot S(t) \cdot I_A\right] = \mathbb{E}\left[\rho(u) \cdot S(u) \cdot I_A\right] \qquad \forall A \in \mathcal{F}_t$$

or equivalent

$$\begin{aligned} \mathbb{E}\left[e^{-\int_0^t r(a)da} \cdot \xi(t) \cdot S(t) \cdot I_A\right] &= \mathbb{E}\left[e^{-\int_0^u r(a)da} \cdot \xi(u) \cdot S(u) \cdot I_A\right] \\ \mathbb{E}^Q\left[e^{-\int_0^t r(a)da} \cdot \xi(T) \cdot S(t) \cdot I_A\right] &= \mathbb{E}^Q\left[e^{-\int_0^u r(a)da} \cdot \xi(T) \cdot S(u) \cdot I_A\right] \end{aligned} \tag{10.1}$$

where formula10.1 follows from

$$e^{-\int_0^t r(a)da} \cdot S(t)$$

being a Q-martingale. Remember:

$$\mathbb{E}\left[e^{-\int_0^t r(a)da} \cdot S(t) \cdot I_A \cdot \mathbb{E}\left[\xi(T) \middle| \mathcal{F}_t\right]\right]$$

The price at time t of a u-maturity discount bond is

$$P(t,u) = \mathbb{E}\left[\frac{\rho(u)}{\rho(t)} \cdot 1 \middle| \mathcal{F}_t\right]$$

which follows from

$$\begin{aligned} \rho(T) \cdot P(t,u) &= \mathbb{E}\left[\rho(u) \cdot P(u,u) \middle| \mathcal{F}_t\right] \\ P(u,u) &= 1 \end{aligned}$$

Now we look at the Y-factors

$$dY = \kappa \cdot (\theta - Y)\, dt + \Sigma dW$$

where κ, θ and Σ are constant and W is a N-vector of independent Brownian motions. The factors are called *gaussian factors* because they are no square-root processes. Further

$$\frac{d\rho}{\rho} = -rdt + \sum_{i=1}^{N} \gamma_i \cdot Y_i dW_i$$

under P (not Q!) and so

$$r(t) = \delta_0 + Y' \cdot \Lambda \cdot Y$$

where Λ is constant and positive semidefinite and δ_0 is positive. So positive short rates are guaranted, while we still have a quite flexible correlation structure.

11 Portfolio consumption choice in complete markets

We need to maximize

$$\mathbb{E}\left[\int_0^T u\left(c_t\right)dt\right]$$

subject to

$$\mathbb{E}\left[\int_0^T \rho\left(t\right)\cdot c\left(t\right)dt\right] = w_0$$

where w_0 denotes the initial wealth. Lagranian give us

$$\mathbb{E}\left[\int_0^T u\left(c_t\right)dt - \lambda\left[\int_0^T \rho\left(t\right)c\left(t\right)dt - w_0\right]\right]$$

So the first order condition is

$$\begin{aligned} u'\left(c_t\right) - \lambda\cdot\rho\left(t\right) &= 0 \qquad \forall t \\ \frac{\rho\left(s\right)}{\rho\left(t\right)} &= \frac{u'\left(c_s\right)}{u'\left(c_t\right)} \\ \rho\left(t\right) &= e^{-\int_0^t r(u)du}\cdot\xi\left(t\right) \\ \frac{d\rho}{\rho} &= -r\left(t\right)dt + \frac{d\xi}{\xi} \end{aligned}$$

where ξ is a P-martingale. So we have

$$\frac{d\rho}{\rho} = -r\left(t\right)dt + \text{stochastic part}$$

and we know that the bond prices are determined by the model of $r(t)$ and the stochastic part of $\frac{d\rho}{\rho}$. Further we calculate (using Ito)

$$\begin{aligned}
\rho(t)\cdot S(t) &= P-\text{martingale}\\
\frac{d(\rho\cdot S)}{\rho\cdot S} &= \frac{d\rho}{\rho}+\frac{dS}{S}+\left(\frac{d\rho}{\rho}\right)\cdot\left(\frac{dS}{S}\right)\\
&= -rdt+\text{stochastic part}+\frac{dS}{S}+\left(\frac{d\rho}{\rho}\right)\cdot\left(\frac{dS}{S}\right)\\
\frac{dS}{S} &= rdt-\left(\frac{d\rho}{\rho}\right)\cdot\left(\frac{dS}{S}\right)-\text{stochastic part}\\
\mathbb{E}\left[\frac{dS}{S}\right] &= rdt-\left(\frac{d\rho}{\rho}\right)\cdot\left(\frac{dS}{S}\right)
\end{aligned}$$

where we call the term $\left(\frac{d\rho}{\rho}\right)\cdot\left(\frac{dS}{S}\right)$ the *risk premium*.

Example

Maximize the portfolio choice

$$\max\mathbb{E}\left[\int_0^T e^{-\delta\cdot t}\cdot u(c_t)\,dt\right]$$

We start with

$$e^{-\delta\cdot t}\cdot u'(c(t))=\lambda\cdot\rho(t)$$

and use Ito to get

$$-\delta dt+\frac{du'(c(t))}{u'(c(t))}=\frac{d\rho}{\rho}$$

Using Ito once again it follows

$$du'(c(t))=u''(c(t))\,dc(t)+\frac{1}{2}\cdot u'''(c(t))\,(dc)^2$$

so the stochastic part of $\frac{d\rho}{\rho}$ is equal to stochastic part of $\frac{u''(c(t))}{u'(c(t))}$ multiplied by the stochastic part of dc. As final result for the risk premium we get

$$-\left(\frac{d\rho}{\rho}\right)\cdot\left(\frac{dS}{S}\right)=\underbrace{-\frac{u''(c(t))\cdot c(t)}{u'(c(t))}}_{\text{coefficient of rel. risk aversion}}\cdot\left(\frac{dc}{c}\right)\cdot\left(\frac{dS}{S}\right)$$

so the risk premium is equal to the coefficient of relative risk aversion mulitplied by the covariance with $\frac{dc}{c}$.